Chapter Summary

In this chapter we have reviewed how to find the distance traveled by an object in motion along a line and (for BC students) along a parametrically defined curve in a plane. We've also looked at a broad variety of applications of the definite integral to other situations where definite integrals of rates of change are used to determine accumulated change, using limits of Riemann sums to create the integrals required.

Practice Exercises

The aim of these questions is mainly to reinforce how to set up definite integrals, rather than how to integrate or evaluate them. Therefore we encourage using a graphing calculator wherever helpful.

#1
$v(t) = s'(t)$
$= 3t^2 - 12t + 9$
$v(t) = 0$
$3(t^2 - 4t + 3) = 0$
$(t-3)(t-1) = 0$
$t = 3 \quad t = 1$
Right | Left | Right

1. A particle moves along a line in such a way that its position at time t is given by $s = t^3 - 6t^2 + 9t + 3$. Its direction of motion changes when

↳ $\boxed{v \text{ changes signs}}$

(A) $t = 1$ only (B) $t = 2$ only (C) $t = 3$ only
(D) $t = 1$ and $t = 3$ (E) $t = 1, 2$, and 3

2. A body moves along a straight line so that its velocity v at time t is given by $v = 4t^3 + 3t^2 + 5$. The distance the body covers from $t = 0$ to $t = 2$ equals $s(t) = \int v(t)$

(A) 34 (B) 55 (C) 24 (D) 44 (E) none of these

#2
$s(t) = \int_0^2 (4t^3 + 3t^2 + 5)\,dt$
$= t^4 + t^3 + 5t \Big|_0^2 = 16 + 8 + 10$
* Note if there were any (−)'s change to (+) b/c distance

3. A particle moves along a line with velocity $v = 3t^2 - 6t$. The total distance traveled from $t = 0$ to $t = 3$ equals

(A) 9 (B) 4 (C) 2 (D) 16 (E) none of these (distance = 8)

#3 Find if particle changes direction
$3t^2 - 6t = 0$
$3t(t-2) = 0$
$t = 0 \quad t = 2$

So we need to set up two integrals & reverse sign on 1st.
$-\int_0^2 (3t^2 - 6t)\,dt + \int_2^3 (3t^2 - 6t)\,dt$
$-[t^3 - 3t^2]_0^2 + [t^3 - 3t^2]_2^3$
$-[(8-12) - (0)] + [(27-27) - (8-12)]$
$4 + 4$

4. The net change in the position of the particle in Question 3 is

(A) 2 (B) 4 (C) 9 (D) 16 (E) none of these $\boxed{\Delta x = 0}$

#4
$\Delta s = \int_0^3 (3t^2 - 6t)\,dt$
* we acct for neg & positive direction for position
$= t^3\Big|_0^3 - 3t^2\Big|_0^3$
$= (27 - 0) - (27 - 0) = 0$

5. The acceleration of a particle moving on a straight line is given by $a = \cos t$, and when $t = 0$ the particle is at rest. The distance it covers from $t = 0$ to $t = 2$ is

$v(0) = 0$

(A) $\sin 2$ (B) $1 - \cos 2$ (C) $\cos 2$ (D) $\sin 2 - 1$ (E) $-\cos 2$

#5
$v(t) = \int \cos t\,dt = \sin t + C$
↳ $v(0) = 0 \quad 0 = \sin 0 + C$ so $\boxed{C = 0}$
$s(t) = \int_0^2 \sin t\,dt$
$= -\cos t\Big|_0^2 = -\cos 2 + \cos 0$

6. During the worst 4-hr period of a hurricane the wind velocity, in miles per hour, is given by $v(t) = 5t - t^2 + 100$, $0 \le t \le 4$. The average wind velocity during this period (in mph) is

Not avg rate

(A) 10 (B) 100 (C) 102 (D) $104\frac{2}{3}$ (E) $108\frac{2}{3}$

#6
$v_{avg} = \dfrac{\int_0^4 v(t)}{4-0} = \dfrac{\int_0^4 5t - t^2 + 100}{4}$
$= \frac{1}{4}\left(\frac{5t^2}{2}\Big|_0^4 - \frac{t^3}{3}\Big|_0^4 + 100t\Big|_0^4\right)$
$= \frac{1}{4}\left(40 - \frac{64}{3} + 400\right)$
$= \frac{1}{4}\left(\frac{120}{3} - \frac{64}{3} + \frac{1200}{3}\right)$
$= \frac{1}{4}\left(\frac{1256}{3}\right)$

7. A car accelerates from 0 to 60 mph in 10 sec, with constant acceleration. (Note that 60 mph = 88 ft/sec.) The acceleration (in ft/sec²) is

(A) 5.3 (B) 6 (C) 8 (D) 8.8 (E) none of these

#7
$a = \dfrac{dv}{dt} = \dfrac{v(10) - v(0)}{10 - 0} = \dfrac{60 - 0}{10} = \dfrac{88 - 0}{10} = 8.8 \text{ ft/sec}^2$

BC ONLY

For Questions 8–10 use the following information: The velocity $\mathbf{v}$ of a particle moving on a curve is given, at time t, by $\mathbf{v} = \langle t, -(1-t)\rangle$. When $t = 0$, the particle is at point (0,1).

8. At time t the position vector $\mathbf{R}$ is

(A) $\left\langle \frac{t^2}{2}, -\frac{(1-t^2)}{2} \right\rangle$ (B) $\left\langle \frac{t^2}{2}, -\frac{(1-t)^2}{2} \right\rangle$

(C) $\left\langle \frac{t^2}{2}, -\frac{t^2-2t}{2} \right\rangle$ (D) $\left\langle \frac{t^2}{2}, -\frac{t^2-2t+2}{2} \right\rangle$

(E) $\left\langle \frac{t^2}{2}, (1-t)^2 \right\rangle$

9. The acceleration vector at time $t = 2$ is

(A) $\langle 1, 1\rangle$ (B) $\langle 1, -1\rangle$ (C) $\langle 1, 2\rangle$ (D) $\langle 2, -1\rangle$ (E) none of these

10. The speed of the particle is at a minimum when t equals

(A) 0 (B) $\frac{1}{2}$ (C) 1 (D) 1.5 (E) 2

11. A particle moves along a curve in such a way that its position vector and velocity vector are perpendicular at all times. If the particle passes through the point (4, 3), then the equation of the curve is

(A) $x^2 + y^2 = 5$ (B) $x^2 + y^2 = 25$ (C) $x^2 + 2y^2 = 34$
(D) $x^2 - y^2 = 7$ (E) $2x^2 - y^2 = 23$

12. The acceleration of an object in motion is given by the vector $\vec{a}(t) = (2t, e^t)$. If the object's initial velocity was $\vec{v}(0) = (2,0)$, which is the velocity vector at any time t?

(A) $\vec{v}(t) = \langle t^2, e^t\rangle$ (B) $\vec{v}(t) = \langle t^2, e^t + 1\rangle$ (C) $\vec{v}(t) = \langle t^2 + 2, e^t\rangle$
(D) $\vec{v}(t) = \langle t^2 + 2, e^t - 1\rangle$ (E) $\vec{v}(t) = \langle 2, e^t - 1\rangle$

13. The velocity of an object is given by $\vec{v}(t) = (3\sqrt{t}, 4)$. If this object is at the origin when $t = 1$, where was it at $t = 0$?

(A) (–3,–4) (B) (–2,–4) (C) (2,4)

(D) $\left(\frac{3}{2}, 0\right)$ (E) $\left(-\frac{3}{2}, 0\right)$

14. Suppose the current world population is 6 billion and the population t years from now is estimated to be $P(t) = 6e^{0.024t}$ billion people. On the basis of this supposition, the $\boxed{\text{average population}}$ of the world, in billions, over the next 25 years will be approximately

(A) 6.75 (B) 7.2 (C) 7.8 (D) 8.2 (E) 9.0

$$\text{Avg} = \frac{\int_0^{25} P(t)\,dt}{25} = \frac{1}{25}\int_0^{25} 6e^{0.024t}\,dt \approx 8.2 \text{ billion ppl}$$

#15) $P(t) = \int_0^t R(x)\,dx$

$P(t) = \int_0^t (10+40t)\,d$

$P(t) = 10x\Big|_0^t + \frac{40t^2}{2}\Big|_0^t$

$P(t) = 10t + 20t^2$ — we want $P(t) = 100$

$100 = 10t + 20t^2$

$0 = 10(2t^2 + t - 10)$ $a \cdot c = -20$, $b = 1$, $4, 5$

$= 10(t-2)(2t+5)$

$(t=2)$ so 2 hours after 8am

let 8 am be $t=0$

15. A beach opens at 8 A.M. and people arrive at a rate of $R(t) = 10 + 40t$ people per hour, where t represents the number of hours the beach has been open. Assuming no one leaves before noon, at what time will there be 100 people there?

(A) 9:45 (B) 10:00 (C) 10:15 (D) 10:30 (E) 10:45

16. A stone is thrown upward from the ground with an initial velocity of 96 ft/sec. Its average velocity (given that $a(t) = -32$ ft/sec²) during the first 2 sec is

(A) 16 ft/sec (B) 32 ft/sec (C) 64 ft/sec
(D) 80 ft/sec (E) 96 ft/sec

Avg $= \frac{\int_0^2 (-32t+96)\,dt}{2-0}$

$= \frac{-16t^2\Big|_0^2 + 96t\Big|_0^2}{2} = \frac{-64+192}{2} = \frac{128}{2}$

#17) Avg $= \frac{\int_0^4 \frac{30}{(t+1)^2}\,dt}{4-0}$ $u = t+1$, $du = dt$

$= \frac{30}{4}\int_0^4 \frac{du}{u^2}$

$= \frac{15}{2}\int_0^4 u^{-2}\,du = \frac{15}{2}\,\frac{u^{-1}}{-1}$

$= -\frac{15}{2(t+1)}\Big|_0^4 = -\frac{15}{10} + \frac{15}{2} = -\frac{15}{10} + \frac{75}{10} = 6\text{mg}$

17. Suppose the amount of a drug in a patient's bloodstream t hr after intravenous administration is $30/(t+1)^2$ mg. The average amount in the bloodstream during the first 4 hr is

(A) 6.0 mg (B) 11.0 mg (C) 16.6 mg
(D) 24.0 mg (E) none of these

#18) # rumors $= \int_7^{14} (t^2 + 10t)\,dt = \frac{t^3}{3}\Big|_7^{14} + 5t^2\Big|_7^{14}$

$= \left[\frac{2744}{3} - \frac{343}{3}\right] + [980 - 245]$

$\left[\frac{2401}{3} + 735\right] = \frac{2401}{3} + \frac{2259}{3}$

$= \frac{4660}{3} \approx 1553$

18. A rumor spreads through a town at the rate of $(t^2 + 10t)$ new people per day. Approximately how many people hear the rumor during the second week after it was first heard? ↳ undo rate → integrate $t=7$ to $t=14$ day

(A) 1535 (B) 1894 (C) 2000
(D) 2219 (E) none of these

#19) $G(t) = \int_0^8 1000e^{-0.3t}\,dt$ $u = -0.3t$, $du = -0.3dt$

$= \frac{1000}{-0.3}\, e^{-0.3t}\Big|_0^8$

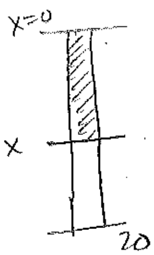
$= 3030.940$

19. Oil is leaking from a tanker at the rate of $1000e^{-0.3t}$ gal/hr, where t is given in hours. The total number of gallons of oil that will leak out during the first 8 hr is approximately undo rate → integrate $t=0$ to $t=8$

(A) 1271 (B) 3031 (C) 3161 (D) 4323 (E) 11,023

Rate $= \frac{\Delta \text{vehicle}}{\Delta \text{mile}}$

20. Assume that the density of vehicles (number per mile) during morning rush hour, for the 20-mi stretch along the New York State Thruway southbound from the Tappan Zee Bridge, is given by $f(x)$, where x is the distance, in miles, south of the bridge. Which of the following gives the number of vehicles (on this 20-mi stretch) from the bridge to a point x mi south of the bridge? ↳ undo rate ↳ integrate



(A) $\int_0^x f(t)\,dt$ (B) $\int_x^{20} f(t)\,dt$ (C) $\int_0^{20} f(x)\,dx$

(D) $\sum_{k=1}^{n} f(x_k)\Delta x$ (where the 20-mi stretch has been partitioned into n equal subintervals)

(E) none of these

21. The center of a city that we will assume is circular is on a straight highway. The radius of the city is 3 mi. The density of the population, in thousands of people per square mile, is given approximately by $f(r) = 12 - 2r$ at a distance r mi from the highway. The population of the city (in 1000s) is given by the integral

(A) $\int_0^3 (12 - 2r)\, dr$ **(B)** $2\int_0^3 (12 - 2r)\sqrt{9 - r^2}\, dr$

(C) $4\int_0^3 (12 - 2r)\sqrt{9 - r^2}\, dr$ **(D)** $\int_0^3 2\pi r(12 - 2r)\, dr$

(E) $2\int_0^3 2\pi r(12 - 2r)\, dr$

22. The population density of Winnipeg, which is located in the middle of the Canadian prairie, drops dramatically as distance from the center of town increases. This is shown in the following table:

x = distance (in mi) from the center	0	2	4	6	8	10
$f(x)$ = density (hundreds of people/mi²)	50	45	40	30	15	5

Using a Riemann sum, we can calculate the population living within a 10-mi radius of the center to be approximately

(A) 608,500 **(B)** 650,000 **(C)** 691,200
(D) 702,000 **(E)** 850,000

23. If a factory continuously dumps pollutants into a river at the $\boxed{\text{rate}}$ of $\frac{\sqrt{t}}{180}$ tons per day, then the amount dumped after 7 weeks is approximately

↳ 49days

↳ undo rate → integrate

(A) 0.07 ton **(B)** 0.90 ton **(C)** 1.55 tons
(D) 1.9 tons **(E)** 1.27 tons

#23

$= \int_0^{49} \frac{\sqrt{t}}{180}\, dt = \frac{1}{180}\int_0^{49} t^{1/2}\, dt$

$= \frac{1}{180} \cdot \frac{2t^{3/2}}{3}\Big|_0^{49} = \frac{1}{180}\left(\frac{2}{3}\sqrt{49^3}\right)$

$= \frac{1}{270}(343) \approx 1.27$ tons

24. A roast at 160°F is put into a refrigerator whose temperature is 45°F. The temperature of the roast is cooling at time t at the $\boxed{\text{rate}}$ of $(-9e^{-0.08t})$°F per minute. The temperature, to the nearest degree F, of the roast 20 min after it is put in the refrigerator is ↳ integrate

(A) 45° **(B)** 70° **(C)** 81° **(D)** 90° **(E)** 115°

#24

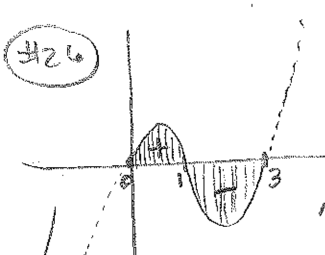
$= \int_0^{20} -9e^{-0.08t}\, dt = -89.7$°F

160°F − 89.7° = 70°F

25. How long will it take to release 9 tons of pollutant if the rate at which pollutant is being released is $te^{-0.3t}$ tons per week?

(A) 10.2 weeks **(B)** 11.0 weeks **(C)** 12.1 weeks
(D) 12.9 weeks **(E)** none of these

#26



26. What is the exact total area bounded by the curve $f(x) = x^3 - 4x^2 + 3x$ and the x-axis?

All positive

(A) −2.25 **(B)** 2.25 **(C)** 3 **(D)** 3.083 **(E)** none of these

Intersection Pts

$x^3 - 4x^2 + 3x = 0$

$x(x^2 - 4x + 3) = 0$

$x(x-1)(x-3)$

$x = 0,\ x = 1,\ x = 3$

↳ $+\int_0^1 (x^3 - 4x^2 + 3x)\, dx - \int_1^3 (x^3 - 4x^2 + 3x)$

0.416667 + 2.66667

3.083

(#27) Calc $= \int_0^3 (-0.1t^2 - 0.3t + 2)\,dt$

$= 3.75$ gal

27. Water is leaking from a tank at the $\boxed{\text{rate}}$ of $(-0.1t^2 - 0.3t + 2)$ gal/hr. The total amount, in gallons, that will leak out in the next 3 hr is approximately

→ integrate

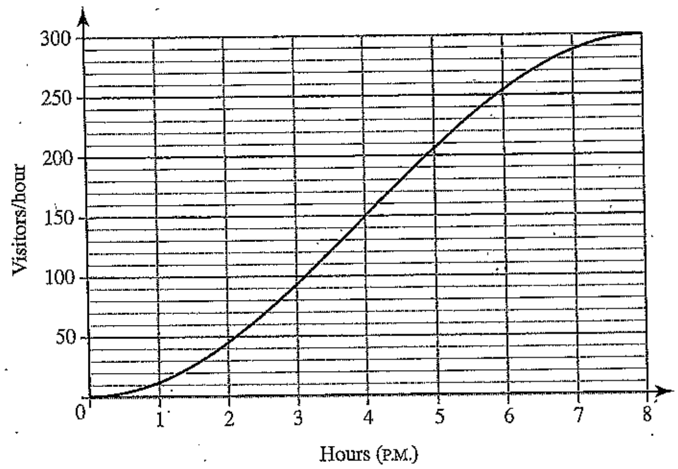
(A) 1.00 (B) 2.08 (C) 3.13 (D) 3.48 (E) 3.75

(#28) 1st hour	2nd hour
$\int_0^1 1000e^{0.03t}\,dt$	$\int_1^2 1000e^{0.03t}\,dt$
1015.15	1046.07

28. A bacterial culture is growing at the $\boxed{\text{rate}}$ of $1000e^{0.03t}$ bacteria in t hr. The total increase in bacterial population during the second hour is approximately

(A) 46 (B) 956 (C) 1046 (D) 1061 (E) 2046

29. A website went live at noon, and the rate of hits (visitors/hour) increased continuously for the first 8 hours, as shown in the graph below.



Approximately when did the 200th visitor go to this site?

(A) before 2 P.M. (B) between 2 and 3 P.M. (C) between 3 and 4 P.M.
(D) between 4 and 5 P.M. (E) after 5 P.M.

(#30) $A_{Trap} = \frac{1}{2} b(h_1 + h_2)$

$A = \frac{1}{2}[2(2+3) + 1(3+1) + 2(1-1) + 2(-1-2) + 3(-2+0)]$

$= \frac{1}{2}[10 + 4 + 0 - 6 - 6]$

$= \frac{1}{2}[2]$

$= 1$ unit

30. An observer recorded the velocity of an object in motion along the x-axis for 10 seconds. Based on the table below, use a trapezoidal approximation to estimate how far from its starting point the object came to rest at the end of this time.

position not distance so neg matters

b= 2 1 2 2 3

t (sec)	0	2	3	5	7	10
$v(t)$ (units/sec) h=	2	3	1	−1	−2	0

(A) 0 units (B) 1 unit (C) 3 units (D) 4 units (E) 6 units

CHALLENGE **31.** An 18-wheeler traveling at speed v mph gets about $(4 + 0.01v)$ mpg (miles per gallon) of diesel fuel. If its speed is $80\frac{t+1}{t+2}$ mph at time t, then the amount, in gallons, of diesel fuel used during the first 2 hr is approximately

(A) 20 (B) 21.5 (C) 23.1 (D) 24 (E) 25

Answer Key

1. D	8. D	15. B	22. C	29. C
2. A	9. A	16. C	23. E	30. B
3. E	10. B	17. A	24. B	31. C
4. E	11. B	18. A	25. A	
5. B	12. D	19. B	26. D	
6. D	13. B	20. A	27. E	
7. D	14. D	21. C	28. C	

Answers Explained

1. **(D)** Velocity $v(t) = \frac{ds}{dt} = 3(t-1)(t-3)$, and changes sign both when $t = 1$ and when $t = 3$.

2. **(A)** Since $v > 0$ for $0 \leqq t \leqq 2$, the distance is equal to $\int_0^2 (4t^3 + 3t^2 + 5)\, dt$.

3. **(E)** The answer is 8. Since the particle reverses direction when $t = 2$, and $v > 0$ for $t > 2$ but $v < 0$ for $t < 2$, therefore, the total distance is

$$-\int_0^2 (3t^2 - 6t)\, dt + \int_2^3 (3t^2 - 6t)\, dt.$$

4. **(E)** $\int_0^3 (3t^2 - 6t)\, dt = 0$, so there is no change in position.

5. **(B)** Since $v = \sin t$ is positive on $0 < t \leqq 2$, the distance covered is $\int_0^2 \sin t\, dt = 1 - \cos 2$.

6. **(D)** Average velocity $= \frac{1}{4-0} \int_0^4 (5t - t^2 + 100)\, dt = 104\frac{2}{3}$ mph.

7. **(D)** The velocity v of the car is linear since its acceleration is constant:

$$a = \frac{dv}{dt} = \frac{(60-0)\text{ mph}}{10\text{ sec}} = \frac{88\text{ ft/sec}}{10\text{ sec}} = 8.8\text{ ft/sec}^2$$

8. **(D)** $\mathbf{v} = \frac{d\mathbf{R}}{dt}$, so $\mathbf{R}(t) = \left\langle \frac{t^2}{2} + c_1, \frac{t^2}{2} - t + c_2 \right\rangle$. Since $\mathbf{R}(0) = \langle 0,1 \rangle$, $c_1 = 0$ and $c_2 = 1$.

9. **(A)** $\mathbf{a} = \mathbf{v}'(t) = \langle 1,1 \rangle$ for all t.

10. **(B)** $\mathbf{v} = \langle t, t-1 \rangle$. $|\mathbf{v}| = \sqrt{t^2 + (t-1)^2}$; $\frac{d|\mathbf{v}|}{dt} = \frac{2t-1}{|\mathbf{v}|}$; $\frac{d|\mathbf{v}|}{dt} = 0$ at $t = \frac{1}{2}$.

Practice Exercises

Part A. Directions: Answer these questions *without* using your calculator.

In Questions 1–10, $a(t)$ denotes the acceleration function, $v(t)$ the velocity function, and $s(t)$ the position or height function at time t. (The acceleration due to gravity is -32 ft/sec^2.)

1. If $a(t) = 4t - 1$ and $v(1) = 3$, then $v(t)$ equals

(A) $2t^2 - t$ (B) $2t^2 - t + 1$ (C) $2t^2 - t + 2$
(D) $2t^2 + 1$ (E) $2t^2 + 2$

#1 $v(t) = \int a(t)dt = \int (4t-1)\,dt$
$= 2t^2 - t + C$ w/ $v(1) = 3$
$3 = 2 - 1 + C$
$c = 2$
$\boxed{v(t) = 2t^2 - t + 2}$

2. If $a(t) = 20t^3 - 6t$, $s(-1) = 2$, and $s(1) = 4$, then $v(t)$ equals

(A) $t^5 - t^3$ (B) $5t^4 - 3t^2 + 1$ (C) $5t^4 - 3t^2 + 3$
(D) $t^5 - t^3 + t + 3$ (E) $t^5 - t^3 + 1$

#2 $v(t) = \int a(t)dt = \int (20t^3 - 6t)dt$
$= 5t^4 - 3t^2 + C_1$
$s(t) = \int 5t^4 - 3t^2 + C_1$
$= t^5 - t^3 + C_1t + C_2 \rightarrow s(-1) = 2$
$2 = (-1)^5 - (-1)^3 - C_1 + C_2$
$2 = -C_1 + C_2$
$\rightarrow s(1) = 4$
$4 = 1^5 - 1^3 + C_1 + C_2$
$4 = C_1 + C_2$
$-C_1 + C_2 = 2$
$C_1 + C_2 = 4$
$2C_2 = 6$
$C_2 = 3$
$C_1 = 1$
$v(t) = 5t^4 - 3t^2 + 1$

3. Given $a(t)$, $s(-1)$, and $s(1)$ as in Question 2, then $s(0)$ equals

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

#3 from #2
$s(t) = t^5 - t^3 + 1t + 3$
so, $s(0) = 0 - 0 + 0 + 3$
$= 3$

4. A stone is thrown straight up from the top of a building with initial velocity 40 ft/sec and hits the ground 4 sec later. The height of the building, in feet, is

(A) 88 (B) 96 (C) 112 (D) 128 (E) 144

5. The maximum height is reached by the stone in Question 4 after

(A) 4/5 sec (B) 4 sec (C) 5/4 sec (D) 5/2 sec (E) 2 sec

6. If a car accelerates from 0 to 60 mph in 10 sec, what distance does it travel in those 10 sec? (Assume the acceleration is constant and note that 60 mph = 88 ft/sec.)

(A) 40 ft (B) 44 ft (C) 88 ft (D) 400 ft (E) 440 ft

7. A stone is thrown at a target so that its velocity after t sec is $(100 - 20t)$ ft/sec. If the stone hits the target in 1 sec, then the distance from the sling to the target is

(A) 80 ft (B) 90 ft (C) 100 ft (D) 110 ft (E) 120 ft

8. What should the initial velocity be if you want a stone to reach a height of 100 ft when you throw it straight up?

(A) 80 ft/sec (B) 92 ft/sec (C) 96 ft/sec
(D) 112 ft/sec (E) none of these

9. If the velocity of a car traveling in a straight line at time t is $v(t)$, then the difference in its odometer readings between times $t = a$ and $t = b$ is

(A) $\int_a^b |v(t)|\,dt$ → speed (doesn't read negative direction) → total distance = abs value

(B) $\int_a^b v(t)\,dt$

(C) the net displacement of the car's position from $t = a$ to $t = b$

(D) the change in the car's position from $t = a$ to $t = b$

(E) none of these

10. If an object is moving up and down along the y-axis with velocity $v(t)$ and $s'(t) = v(t)$, then it is false that $\int_a^b v(t)\,dt$ gives

(A) $s(b) - s(a)$

(B) the net distance traveled by the object between $t = a$ and $t = b$

(C) the total change in $s(t)$ between $t = a$ and $t = b$

(D) the shift in the object's position from $t = a$ to $t = b$

(E) the total distance covered by the object from $t = a$ to $t = b$

#11 $\int y\,dy = \int x\,dx$

$\frac{y^2}{2} = \frac{x^2}{2} + C$

$y^2 = x^2 + C$

11. Solutions of the differential equation $y\,dy = x\,dx$ are of the form

(A) $x^2 - y^2 = C$ **(B)** $x^2 + y^2 = C$ **(C)** $y^2 = Cx^2$

(D) $x^2 - Cy^2 = 0$ **(E)** $x^2 = C - y^2$

12. Find the domain of the particular solution to the differential equation in Question 11 that passes through point $(-2, 1)$.

(A) $x < 0$ **(B)** $-2 \le x < 0$ **(C)** $x < -\sqrt{3}$

(D) $|x| < \sqrt{3}$ **(E)** $|x| > \sqrt{3}$

#13 Separate variable

$\int \frac{dy}{y} = \int \frac{1}{2\sqrt{x}} \cdot dx$

$\int \frac{1}{y}\,dy = \frac{1}{2}\int x^{-1/2}\,dx$

$\ln|y| = \frac{1}{2}\frac{2x^{1/2}}{1} + C$

$\ln y = \sqrt{x} + C$

13. If $\frac{dy}{dx} = \frac{y}{2\sqrt{x}}$ and $y = 1$ when $x = 4$, then

(A) $y^2 = 4\sqrt{x} - 7$ **(B)** $\ln y = 4\sqrt{x} - 8$ **(C)** $\ln y = \sqrt{x - 2}$

(D) $y = e^{\sqrt{x}}$ **(E)** $y = e^{\sqrt{x}-2}$

$y = Ce^{\sqrt{x}}\Big|_{(4,1)} \rightarrow 1 = Ce^{\sqrt{4}} \Rightarrow C = \frac{1}{e^2} \Rightarrow y = \frac{1}{e^2}e^{\sqrt{x}} = e^{-2} \cdot e^{\sqrt{x}} = e^{\sqrt{x}-2}$

#14 Separate Variables

$\int \frac{dy}{e^y} = \int dx$

$\int e^{-y}\,dy = \int dx$

$-e^{-y} = x + C$

$\ln e^{-y} = \ln|-x + C|$

$-y = \ln|-x+c|$

$y = -\ln|-x+c|$

$y = -\ln|-x+2|$

14. If $\frac{dy}{dx} = e^y$ and $y = 0$ when $x = 1$, then

(A) $y = \ln|x|$ **(B)** $y = \ln|2 - x|$ **(C)** $e^{-y} = 2 - x$

(D) $y = -\ln|x|$ **(E)** $e^{-y} = x - 2$

15. If $\frac{dy}{dx} = \frac{x}{\sqrt{9 + x^2}}$ and $y = 5$ when $x = 4$, then y equals

(A) $\sqrt{9 + x^2} - 5$ **(B)** $\sqrt{9 + x^2}$ **(C)** $2\sqrt{9 + x^2} - 5$

(D) $\frac{\sqrt{9 + x^2} + 5}{2}$ **(E)** none of these

$(1,0) \longrightarrow 0 = -\ln|-1+c| \longrightarrow e^0 = e^{\ln|-1+c|} \rightarrow 1 = -1 + c$ $(c = 2)$

Separate Variable

16. The general solution of the differential equation $x\,dy = y\,dx$ is a family of

(A) circles (B) hyperbolas (C) parallel lines
(D) parabolas (E) lines passing through the origin

#16
$\frac{dy}{y} = \frac{dx}{x}$
$e^{\ln y} = e^{\ln x + C}$
$y = Ce^{\ln x}$
$y = Cx$ line b=0

17. The general solution of the differential equation $\frac{dy}{dx} = y$ is a family of

(A) parabolas (B) straight lines (C) hyperbolas
(D) ellipses (E) none of these

#17 sep. var.
$\int \frac{dy}{y} = \int dx$
$e^{\ln y} = e^{x + C}$
$y = Ce^{x}$ exponential

18. A function $f(x)$ that satisfies the equations $f(x)f'(x) = x$ and $f(0) = 1$ is

(A) $f(x) = \sqrt{x^2 + 1}$ (B) $f(x) = \sqrt{1 - x^2}$ (C) $f(x) = x$
(D) $f(x) = e^x$ (E) none of these

19. The curve that passes through the point (1, 1) and whose slope at any point (x, y) is equal to $\frac{3y}{x}$ has the equation

(A) $3x - 2 = y$ (B) $y^3 = x$ (C) $y = x^3$
(D) $3y^2 = x^2 + 2$ (E) $3y^2 - 2x = 1$

#19
$\frac{dy}{dx} = \frac{3y}{x}$ sep. vars
$\int \frac{dy}{3y} = \int \frac{dx}{x}$
$\frac{1}{3}\ln|y| = \ln|x| + C$
$e^{\ln|y|} = e^{3\ln|x| + C}$
$Y = Ce^{3\ln|x|}$
$Y = C\cancel{e}^{\ln x^3}$
$Y = C \cdot x^3 \Big|_{(1,1)}$
$\hookrightarrow C = 1$

20. What is the domain of the particular solution in Question 19?

(A) all real numbers (B) $|x| \le 1$ (C) $x \neq 0$
(D) $x < 0$ (E) $x > 0$

21. If $\frac{dy}{dx} = \frac{k}{x}$, k a constant, and if $y = 2$ when $x = 1$ and $y = 4$ when $x = e$, then, when $x = 2$, y equals

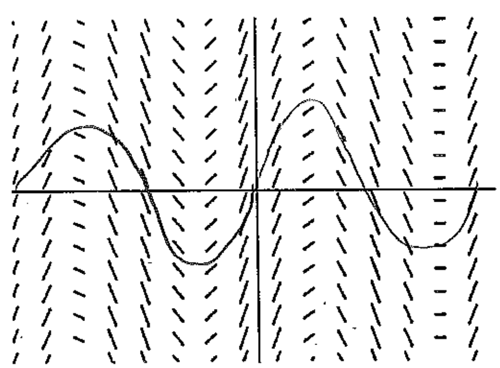
(A) 2 (B) 4 (C) ln 8 (D) ln 2 + 2 (E) ln 4 + 2

#21
$\frac{dy}{dx} = \frac{k}{x}$
$\int dy = k\int \frac{dx}{x}$
$Y = k\ln x + C$
pt(1,2) $2 = k\ln(1) + C$
$C = 2$
pt(e,4) $4 = k\ln e + 2$
$4 = k + 2$
$k = 2$
x=2 $Y = 2\ln 2 + 2$
$= \ln 2^2 + 2$
$= \ln 4 + 2$

22. The slope field shown at the right is for the differential equation

$\frac{dy}{dx}$
(A) $\int y' = \int x + 1$ $y = \frac{x^2}{2} + x + C$
(B) $\int y' = \int \sin x$ $y = -\cos x + C$
(C) $\int y' = \int -\sin x$ $y = \cos x + C$
(D) $\int y' = \int \cos x$ $\boxed{y = \sin x + C}$
(E) $\int y' = \int -\cos x$ $y = -\sin x + C$

★ Soln y looks like a sin cure

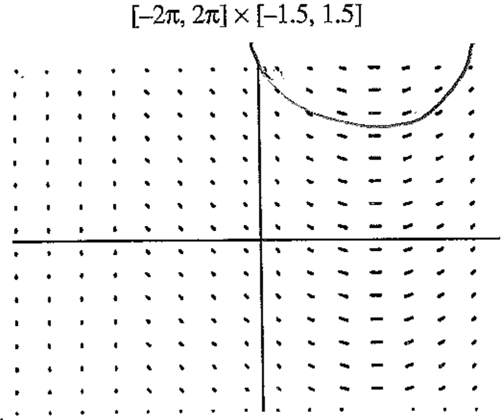


$[-2\pi, 2\pi] \times [-1.5, 1.5]$

23. The slope field at the right is for the differential equation

(A) $\int y' = \int 2x$ $\frac{2x^3}{3} + C$
(B) $\int y' = \int 2x - 4$ $x^2 - 4x + C$
(C) $\int y' = \int 4 - 2x$ $4x - x^2 + C$
(D) $\int y' = \int y$ $\frac{dy}{y} = dx$ $\ln y = x + C$ $y = Ce^x$
(E) $\int y' = \int x + y$?

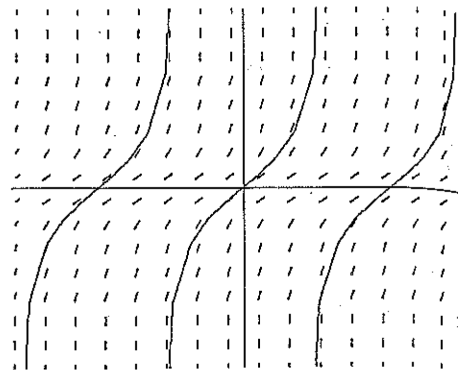
★ Sol'n y looks like $x^2 +$ ___ b/c opens up



$[-4, 4] \times [-12, 12]$

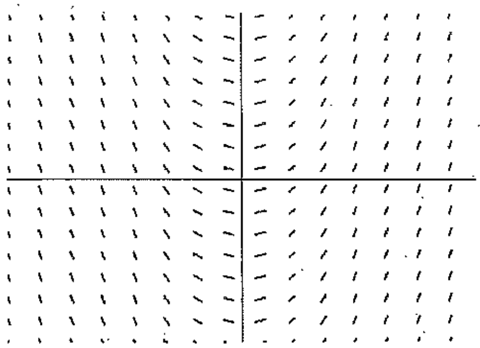
24. A solution curve has been superimposed on the slope field shown at the right. The solution is for the differential equation and initial condition

(A) $y' = \tan x$; $y(0) = 0$
(B) $y' = \cot x$, $y(\pi/4) = 1$
(C) $y' = 1 + x^2$; $y(0) = 0$
(D) $y' = \dfrac{1}{1+x^2}$; $y\left(\dfrac{\pi}{4}\right) = 1$
(E) $y' = 1 + y^2$; $y(0) = 0$

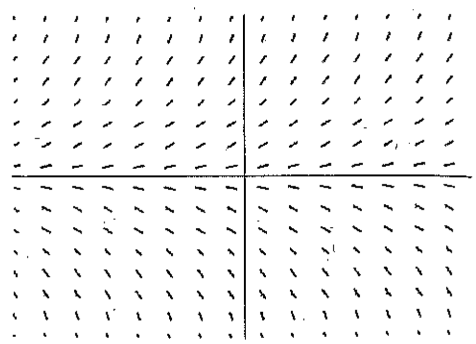


$[-4, 4] \times [-4, 4]$

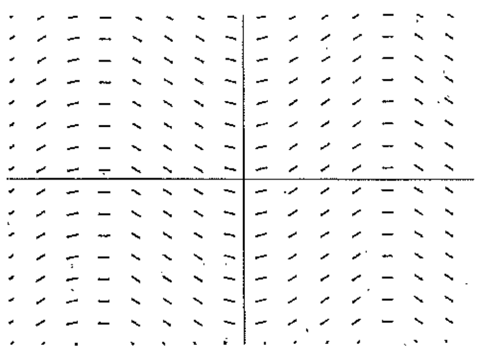
The slope fields below are for Questions 25–30.



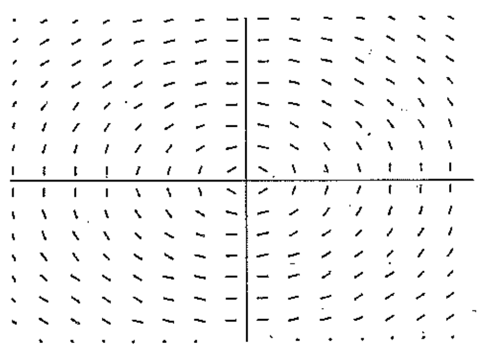
I $[-3, 3] \times [-3, 3]$



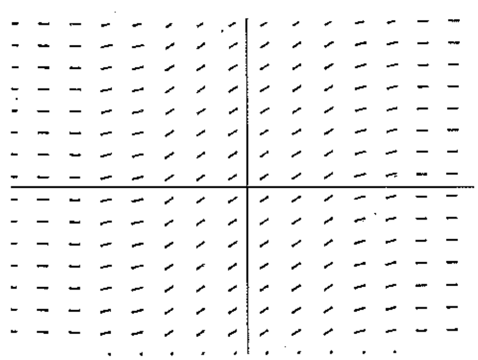
II $[-3, 3] \times [-3, 3]$



III $[-5, 5] \times [-5, 5]$



IV $[-3, 3] \times [-3, 3]$



V $[-2, 2] \times [-2, 2]$

25. Which slope field is for the differential equation $y' = y$?

(A) I (B) II (C) III (D) IV (E) V

#25 $\frac{dy}{dx} = y$ so $\int \frac{dy}{y} = \int dx$
$\ln y = x + C$
$\boxed{y = Ce^x}$ II

26. Which slope field is for the differential equation $y' = -\frac{x}{y}$?

(A) I (B) II (C) III (D) IV (E) V

#26 $\frac{dy}{dx} = -\frac{x}{y}$ so $\int y\,dy = \int -x\,dx$
$\frac{y^2}{2} = -\frac{x^2}{2} + C$ IV
$\boxed{y = \pm\sqrt{C - x^2}}$

27. Which slope field is for the differential equation $y' = \sin x$?

(A) I (B) II (C) III (D) IV (E) V

#27 $\frac{dy}{dx} = \sin x$ so $\int dy = \int \sin x\,dx$
$y = -\cos x + C$ III

28. Which slope field is for the differential equation $y' = 2x$?

(A) I (B) II (C) III (D) IV (E) V

#28 $\frac{dy}{dx} = 2x$ so $\int dy = \int 2x\,dx$
$y = x^2 + C$ I

29. Which slope field is for the differential equation $y' = e^{-x^2}$?

(A) I (B) II (C) III (D) IV (E) V

#29 $\frac{dy}{dx} = e^{-x^2}$ so $\int dy = \int e^{-x^2}dx$
V only one left

30. A particular solution curve of a differential equation whose slope field is shown above in II passes through the point (0,–1). The equation is

(A) $y = -e^x$ (B) $y = -e^{-x}$ (C) $y = x^2 - 1$ (D) $y = -\cos x$
(E) $y = -\sqrt{1 - x^2}$

#30 Refer to #25 II
$y = Ce^x \big|_{(0,-1)}$
$-1 = Ce^0$
$C = -1$
$y = -e^x$

BC ONLY

31. If you use Euler's method with $\Delta x = 0.1$ for the d.e. $y' = x$, with initial value $y(1) = 5$, then, when $x = 1.2$, y is approximately

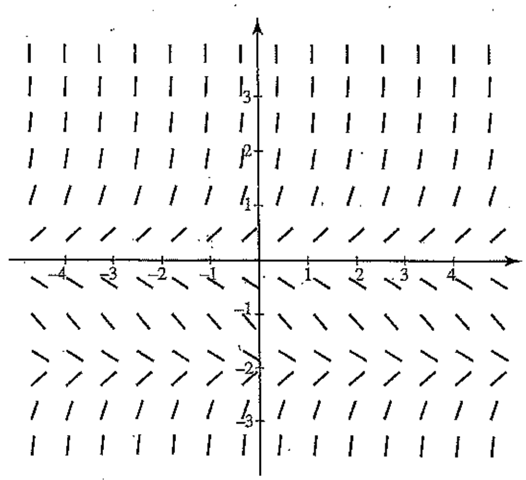
(A) 5.10 (B) 5.20 (C) 5.21 (D) 6.05 (E) 7.10

32. The error in using Euler's method in Question 31 is

(A) 0.005 (B) 0.010 (C) 0.050 (D) 0.500 (E) 0.720

33. Which differential equation has the slope field shown?

(A) $y' = y(y + 2) = y^2 + 2y$ ✓
(B) $y' = x(y + 2) = xy + 2x$
(C) $y' = xy + 2$
(D) $y' = \frac{x}{y + 2}$
(E) $y' = \frac{y}{y + 2}$

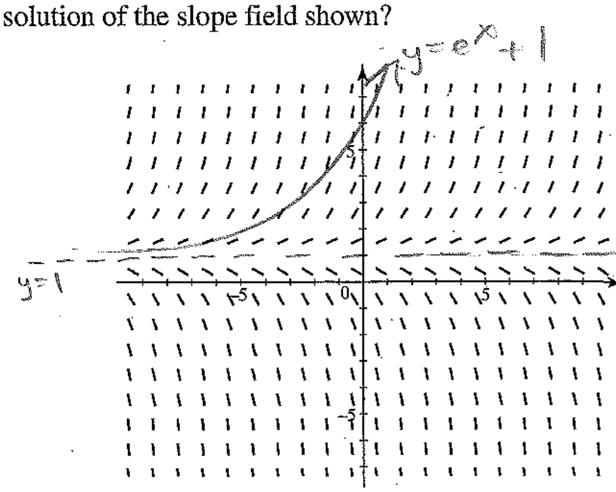


$\leftarrow \frac{dy}{dx} = 0$ @ $y = 0$

$\leftarrow \frac{dy}{dx} = 0$ @ $y = -2$

34. Which function is a possible solution of the slope field shown?

(A) $y = 1 - \frac{1}{x}$
(B) $y = 1 - \ln x$
(C) $y = 1 + \ln x$
(D) $y = 1 + e^x$
(E) $y = 1 + \tan x$



Sep. Vari

#35 $\frac{ds}{dt} = \sin^2\left(\frac{\pi}{2}s\right)$

$\int \frac{ds}{\sin^2(\frac{\pi}{2}s)} = \int dt$

$\int \csc^2(\pi/2\, s)\, ds = \int dt$

$u = \pi/2\, s$
$du = \pi/2\, ds$

$-\frac{2}{\pi}\cot(\pi/2\, s) = t + C$

$-\frac{2}{\pi}\cot\left(\frac{\pi}{2}\right) = 0 + C$

$(0,1)$

$\frac{x}{y}$

$-\frac{2}{\pi}\left(\frac{0}{1}\right) = C$ so $C = 0$

$-\frac{2}{\pi}\cot\left(\frac{3\pi}{4}\right) = t$

$-\frac{2}{\pi}\left(\frac{-\sqrt{2}/2}{\sqrt{2}/2}\right) = t$

$-\frac{2}{\pi}(-1) = t$

$t = \frac{2}{\pi}$

Part B. Directions: Some of the following questions require the use of a graphing calculator.

35. If $\frac{ds}{dt} = \sin^2\left(\frac{\pi}{2}s\right)$ and if $s = 1$ when $t = 0$, then, when $s = \frac{3}{2}$, t is equal to

(A) $\frac{1}{2}$ (B) $\frac{\pi}{2}$ (C) 1 (D) $\frac{2}{\pi}$ (E) $-\frac{2}{\pi}$

36. If radium decomposes at a rate proportional to the amount present, then the amount R left after t yr, if R_0 is present initially and c is the negative constant of proportionality, is given by

(A) $R = R_0 ct$ (B) $R = R_0 e^{ct}$ (C) $R = R_0 + \frac{1}{2}ct^2$
(D) $R = e^{R_0 ct}$ (E) $R = e^{R_0 + ct}$

$P_0 = Ce^{kt}$

37. The population of a city increases continuously at a rate proportional, at any time, to the population at that time. The population doubles in 50 yr. After 75 yr the ratio of the population P to the initial population P_0 is

(A) $\frac{9}{4}$ (B) $\frac{5}{2}$ (C) $\frac{4}{1}$ (D) $\frac{2\sqrt{2}}{1}$ (E) none of these

$\frac{P}{P_0} = 8$

① $P_0 = Ce^{k(0)} \rightarrow C = P_0$

② $2P_0 = \cancel{P_0} e^{50k} \rightarrow k = \frac{\ln 2}{50}$

③ $P_{75} = P_0 e^{75(\frac{\ln 2}{50})}$

$P_{75} = P_0 e^{\frac{3}{2}\ln 2}$

$P_{75} = P_0 \cancel{e}^{\ln 2^{3/2}}$

$P_{75} = P_0(\sqrt{2^3})$

$P_{75} = 8P_0$

38. If a substance decomposes at a rate proportional to the amount of the substance present, and if the amount decreases from 40 g to 10 g in 2 hr, then the constant of proportionality is

(A) $-\ln 2$ (B) $-\frac{1}{2}$ (C) $-\frac{1}{4}$ (D) $\ln\frac{1}{4}$ (E) $\ln\frac{1}{8}$

39. If $(g'(x))^2 = g(x)$ for all real x and $g(0) = 0$, $g(4) = 4$, then $g(1)$ equals

(A) $\frac{1}{4}$ (B) $\frac{1}{2}$ (C) 1 (D) 2 (E) 4

40. The solution curve of $y' = y$ that passes through point (2, 3) is

(A) $y = e^x + 3$ (B) $y = \sqrt{2x+5}$ (C) $y = 0.406e^x$
(D) $y = e^x - (e^2 + 3)$ (E) $y = e^x/(0.406)$

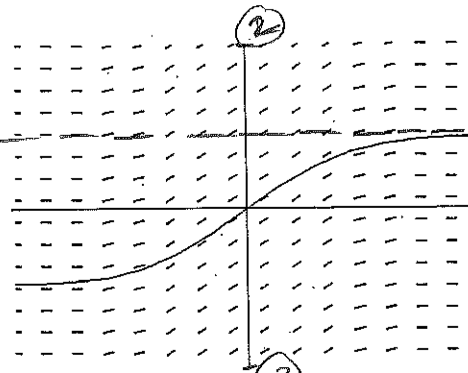
#40 $\frac{dy}{dx} = y$ so $\int \frac{dy}{y} = \int dx$
$\ln|y| = x + C$
$y = Ce^x \Big|_{(2,3)}$
$\hookrightarrow C = \frac{3}{e^2}$
$y = \frac{3}{e^2} e^x$

41. At any point of intersection of a solution curve of the d.e. $y' = x + y$ and the line $x + y = 0$, the function y at that point

(A) is equal to 0 (B) is a local maximum (C) is a local minimum
(D) has a point of inflection (E) has a discontinuity

42. The slope field for $F'(x) = e^{-x^2}$ is shown at the right with the particular solution $F(0) = 0$ superimposed. With a graphing calculator, $\lim_{x\to\infty} F(x)$ to three decimal places is

(A) 0.886 (B) 0.987
(C) 1.000 (D) 1.414
(E) ∞



[−2, 2] × [−2, 2]

approaches close to 1 as $x \to \infty$

$\int_0^x e^{-x^2}$ let $x = 100$

$= 0.886$

BC ONLY

43. The graph displays logistic growth for a frog population F. Which differential equation could be the appropriate model?

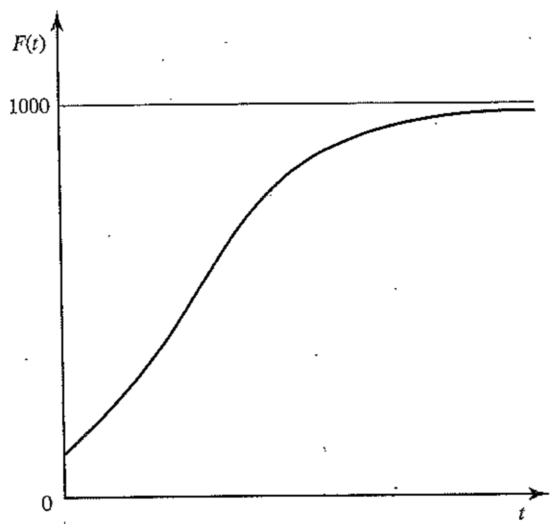
(A) $\frac{dF}{dt} = 1.5F - 0.003F^2$

(B) $\frac{dF}{dt} = 1.5F^2 - 0.003F$

(C) $\frac{dF}{dt} = 3F - 0.003F^2$

(D) $\frac{dF}{dt} = 3F^2 - 0.003F$

(E) $\frac{dF}{dt} = 0.003F^2 - 3F$



44. The table shows selected values of the derivative for a differentiable function f.

x	2	3	4	5	6	7
$f'(x)$	2.0	2.5	1.0	−0.5	−1.5	0.5

Given that $f(3) = 100$, use Euler's method with a step size of 2 to estimate $f(7)$.
(A) 101.5 (B) 102.5 (C) 103 (D) 104 (E) 104.5

#45 $\int \frac{dy}{y-68} = \int -0.11\,dx$

$\ln|y-68| = -0.11t + C$

$y - 68 = Ce^{-0.11t}$

$\boxed{y = Ce^{-0.11t} + 68}$

↳ $y(0) = 180$

$180 = Ce^{0} + 68$

$C = 112$

$y(t) = 112e^{-0.11t} + 68$

$y(10) = 112e^{-0.11(10)} + 68$

$\boxed{= 105°F}$

$y(0) = 180$

45. A cup of coffee at temperature 180°F is placed on a table in a room at 68°F. The d.e. for its temperature at time t is $\frac{dy}{dt} = -0.11(y - 68)$; $y(0) = 180$. After 10 min the temperature (in °F) of the coffee is $t = 0$ to 10 **51.**

(A) 96   (B) 100   (C) 105   (D) 110   (E) 115

46. Approximately how long does it take the temperature of the coffee in Question 45 to drop to 75°F?

(A) 10 min   (B) 15 min   (C) 18 min   (D) 20 min   (E) 25 min

47. The concentration of a medication injected into the bloodstream drops at a rate proportional to the existing concentration. If the factor of proportionality is 30% per hour, in how many hours will the concentration be one-tenth of the initial concentration?   **52.**

(A) 3   (B) $4\frac{1}{3}$ (C) $6\frac{2}{3}$

(D) $7\frac{2}{3}$ (E) none of these

BC ONLY

#46 $y(t) = 75°F$

$y(t) = 112e^{-0.11t} + 68$

$75 = 112e^{-0.11t} + 68$

$7 = 112e^{-0.11t}$

$\ln 0.0625 = \cancel{\ln} e^{-0.11t}$

$\ln|0.0625| = -0.11t$

$\frac{\ln(0.0625)}{-0.11} = t$

$25.205\text{ min} = t$

48. Which of the following statements characterize(s) the logistic growth of a population whose limiting value is L?

I. The rate of growth increases at first.

II. The growth rate attains a maximum when the population equals $\frac{L}{2}$.

III. The growth rate approaches 0 as the population approaches L.

(A) I only   (B) II only   (C) I and II only
(D) II and III only   (E) I, II, and III

49. Which of the following d.e.'s is not logistic?

(A) $P' = P - P^2$ (B) $\frac{dy}{dt} = 0.01y(100 - y)$

(C) $\frac{dx}{dt} = 0.8x - 0.004x^2$ (D) $\frac{dR}{dt} = 0.16(350 - R)$

(E) $f'(t) = kf(t) \cdot [A - f(t)]$ (where k and A are constants)

50. Suppose $P(t)$ denotes the size of an animal population at time t and its growth is described by the d.e. $\frac{dP}{dt} = 0.002P(1000 - P)$. The population is growing fastest

(A) initially   (B) when $P = 500$ (C) when $P = 1000$

(D) when $\frac{dP}{dt} = 0$ (E) when $\frac{d^2P}{dt^2} > 0$

51. According to Newton's law of cooling, the temperature of an object decreases at a rate proportional to the difference between its temperature and that of the surrounding air. Suppose a corpse at a temperature of 32°C arrives at a mortuary where the temperature is kept at 10°C. Then the differential equation satisfied by the temperature T of the corpse t hr later is

(A) $\frac{dT}{dt} = -k(T-10)$ (B) $\frac{dT}{dt} = k(T-32)$ (C) $\frac{dT}{dt} = 32e^{-kt}$

(D) $\frac{dT}{dt} = -kT(T-10)$ (E) $\frac{dT}{dt} = kT(T-32)$

52. If the corpse in Question 51 cools to 27°C in 1 hr, then its temperature (in °C) is given by the equation

(A) $T = 22e^{0.205t}$ (B) $T = 10e^{1.163t}$ (C) $T = 10 + 22e^{-0.258t}$

(D) $T = 32e^{-0.169t}$ (E) $T = 32 - 10e^{-0.093t}$

#51 $\frac{dT}{dt} = \overset{\text{decreases}}{-}k(T - T_A)$

T = temp
t = time

#52 $\frac{dT}{dt} = -k(T - 10)$

$T_A = 10°C$ b/c morque's ambient environment.

separate variables

$\int \frac{dT}{T-10} = \int k\,dt$

$\ln|T-10| = -kt + C$

$T - 10 = Ce^{-kt}$

$T = Ce^{-kt} + 10$ → $T(0) = 32°$ (time = 0)

$32 = Ce^{0} + 10$

$C = 22$

AND $T(1) = 27°$

$27 = 22e^{-k(1)} + 10$

$17 = 22e^{-k}$

$\ln \frac{17}{22} = \ln e^{-k}$

$\ln\left|\frac{17}{22}\right| = -k$

$k = -\ln\left(\frac{17}{22}\right)$

$\boxed{T = 22e^{-\ln(17/22)} + 10}$

Answer Key

1. C	**12.** C	**23.** B	**34.** D	**45.** C
2. B	**13.** E	**24.** E	**35.** D	**46.** E
3. D	**14.** C	**25.** B	**36.** B	**47.** D
4. B	**15.** B	**26.** D	**37.** D	**48.** E
5. C	**16.** E	**27.** C	**38.** A	**49.** D
6. E	**17.** E	**28.** A	**39.** A	**50.** B
7. B	**18.** A	**29.** E	**40.** C	**51.** A
8. A	**19.** C	**30.** A	**41.** C	**52.** C
9. A	**20.** E	**31.** C	**42.** A	
10. E	**21.** E	**32.** B	**43.** C	
11. A	**22.** D	**33.** A	**44.** D	

Answers Explained

1. (C) $v(t) = 2t^2 - t + C$; $v(1) = 3$; so $C = 2$.

2. (B) If $a(t) = 20t^3 - 6t$, then

$$v(t) = 5t^4 - 3t^2 + C_1,$$
$$s(t) = t^5 - t^3 + C_1t + C_2,$$

Since

$$s(-1) = -1 + 1 - C_1 + C_2 = 2$$

and

$$s(1) = 1 - 1 + C_1 + C_2 = 4,$$

therefore

$$2C_2 = 6,\ C_2 = 3,$$
$$C_1 = 1.$$

So

$$v(t) = 5t^4 - 3t^2 + 1.$$

3. (D) From Answer 2, $s(t) = t^5 - t^3 + t + 3$, so $s(0) = C_2 = 3$.

4. (B) Since $a(t) = -32$, $v(t) = -32t + 40$, and the height of the stone $s(t) = -16t^2 + 40t + C$. When the stone hits the ground, 4 sec later, $s(t) = 0$, so

$$0 = -16(16) + 40(4) + C,$$
$$C = 96 \text{ ft}.$$

5. (C) From Answer 4

$$s(t) = -16t^2 + 40t + 96.$$

Then

$$s'(t) = -32t + 40,$$

which is zero if $t = 5/4$, and that yields maximum height, since $s''(t) = -32$.

6. (E) The velocity $v(t)$ of the car is linear, since its acceleration is constant and

$$a(t) = \frac{dv}{dt} = \frac{(60 - 0) \text{ mph}}{10 \text{ sec}} = \frac{88 \text{ ft/sec}}{10 \text{ sec}} = 8.8 \text{ ft/sec}^2$$

$$v(t) = 8.8t + C_1 \quad \text{and} \quad v(0) = 0, \quad \text{so } C_1 = 0;$$
$$s(t) = 4.4t^2 + C_2 \quad \text{and} \quad s(0) = 0, \quad \text{so } C_2 = 0;$$
$$s(10) = 4.4(10^2) = 440 \text{ ft}.$$

7. (B) Since $v = 100 - 20t$, $s = 100t - 10t^2 + C$ with $s(0) = 0$. So $s(1) = 100 - 10 = 90$ ft.